On Plane and Space Curves

Mentor: Daniel Restrepo Montoya

Student: Daniel Naves

What is a curve?

Let $I \subset \mathbb{R}$ be an open interval, not necessarily bounded. A function $\alpha : I \to \mathbb{R}^3$ is a curve if it is a C^∞ map, i.e., if it has derivatives of all orders on I.
We say that α is a *plane curve* if there exists a plane $P \subset \mathbb{R}^3$ such that $\alpha(I) \subset P$.
A *space curve* is a curve whose points do not necessarily all lie on a single plane.

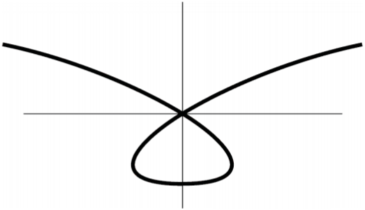

Figure 1: A plane curve with a self-intersection

Arc Length

One may seek to study the length of the trace of α over some compact interval $[a, b] \subset I$. A natural approach is to define a partition $P = \{a = t_0 < t_1 < ... < t_n = b\}$ of $[a, b]$ and approximate the length of $\alpha([a, b])$ with the sum

$$\sum_{i=1}^{n} |\alpha(t_i) - \alpha(t_{i-1})|.$$

It can be shown that if $|P| = \max_{1 \leq i \leq n}\{|t_i - t_{i-1}|\}$, then

$$\lim_{|P| \to 0} = \int_a^b |\alpha'(t)| dt,$$

which suggests the definition of $\int_a^b |\alpha'(t)| dt$ as *the length of* α *from* a *to* b.

Parametrization by Arc Length

We say that α is a *regular curve* if $\alpha'(t) \neq 0$ for each $t \in I$, i.e., if the tangent line of α is well-defined at each of its points.
For any $t_0 \in I$, we can define the *arc length function from* t_0, given by

$$S(t) = \int_{t_0}^{t} |\alpha'(u)| du.$$

We say that α is parametrized by arc length (p.b.a.l) if $|\alpha'(t)| = 1$ for each $t \in I$, so that $S(t) = t - t_0$. . Since $S'(t) = |\alpha'(t)|$, the inverse function theorem tells us that if α is regular, then S is increasing and open. Hence if $J = S(I)$, then $S : I \to J$ is a diffeomorphism between open intervals.

Let $\beta : J \to \mathbb{R}^3$ be the reparametrization of α given by $\beta = \alpha \circ S^{-1}$. It follows that

$$\beta'(s) = \alpha'(S^{-1}(s))(S^{-1})'(s) = \frac{\alpha'(S^{-1}(s))}{|\alpha'(S^{-1}(s))|},$$

so that $|\beta'(s)| = 1$ for each $s \in J$. Therefore, any regular curve admits a reparametrization by arc length.
Let $\alpha : I \to \mathbb{R}^2$ be a curve p.b.a.l, and let the tangent unit vector $\alpha'(s)$ be denoted by $T(s)$. Let the *normal vector of* α *at* s be given by $N(s) = JT(s)$, where J is the linear transformation corresponding to a 90-degree counter-clockwise rotation. From the equalities $|T(s)|^2 = |N(s)|^2 = 1$ and $\langle T(s), N(s)\rangle = 0$, we can conclude, by taking derivatives, that

$$\langle T'(s), T(s)\rangle = \langle N'(s), N(s)\rangle = \langle T'(s), N(s)\rangle + \langle T(s), N'(s)\rangle = 0.$$

Observe that $T'(s)$ has the same direction as $N(s)$, so that, for each $s \in I$, we have that $T'(s) = k(s)N(s)$ for some $k(s) \in \mathbb{R}$. The number $k(s)$ is called the *curvature of* α *at* s.
We have constructed, for each $s \in I$, a positively-oriented orthonormal basis $\{T(s), N(s)\}$ for $\mathbb{R}^2$. This pair of vectors is called the *oriented Frenet dihedron* of α at s. The equations

$$T'(s) = k(s)N(s) \text{ and } N'(s) = -k(s)T(s)$$

are called the *Frenet equations of the curve* α.

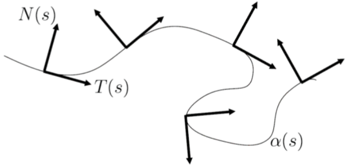

Figure 2: The evolution of the Frenet dihedrons unveil the geometrical features of the curve

Fundamental Theorem of the Local Theory of Plane Curves

Let $k_0 : I \to \mathbb{R}$ be a differentiable function defined on an open interval $I \subset \mathbb{R}$. Then, there exists a plane curve $\alpha : I \to \mathbb{R}^2$ p.b.a.l. such that $k_\alpha(s) = k_0(s)$ for every $s \in I$, where k_α is the curvature function of α. Moreover, if $\beta : I \to \mathbb{R}^2$ is another plane curve p.b.a.l. with $k_\beta(s) = k_0(s)$, for all $s \in I$, then there exists a direct rigid motion $M : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\beta = M \circ \alpha$.

Space curves

We can, as we did for plane curves, associate a positively oriented, orthonormal basis of $\mathbb{R}^3$ to each point of a space curve.
Let $\alpha : I \to \mathbb{R}^3$ be a curve p.b.a.l and $T(s) = \alpha'(s)$. Since $|T(s)|^2 = 1$ for each $s \in I$, it follows that $\langle T'(s), T(s) \rangle = 0$ for each $s \in I$. We now define the curvature k of α at s by $k(s) = |T'(s)|$. Unlike with plane curves, the definition forces the curvature to be non-negative at each $s \in I$.
Assume that k is strictly positive. Then one may consider the vector

$$N(s) = \frac{T'(s)}{|T'(s)|} = \frac{1}{k(s)} T'(s).$$

From previous reasoning, we see that $\langle T(s), N(s) \rangle = 0$ for each $s \in I$. Hence, in order to construct our orthonormal basis, it suffices to define a third vector $B(s) = T(s) \times N(s)$, where $\times$ denotes the vector product of Euclidean three-space. We call this vector *the binormal vector of the curve α at s.*

Torsion

From the definition of B, we can conclude that

$$B(s) = T'(s) \times N(s) + T(s) \times N'(s) = T(s) \times N'(s)$$

since $T'(s)$ and $N(s)$ have the same direction. Hence

$$\langle B'(s), T(s) \rangle = \det(T(s), N'(s), T(s)) = 0.$$

Furthermore, $|B(s)|^2 = 1$, so that $\langle B'(s), B(s) \rangle = 0$. This tells us that $B'(s)$ has no components in the directions of $T(s)$ or $B(s)$. It follows that

$$B'(s) = \tau(s) N(s),$$

with $\tau(s) \in \mathbb{R}$ for each $s \in I$. The number $\tau(s)$ is called the *torsion of the curve α at s.*

Finally, we study the derivative of the normal vector: From the equalities $|N(s)|^2 = 1$, $\langle N(s), T(s)\rangle = 0$, and $\langle N(s), B(s)\rangle = 0$, it follows that

$$\begin{aligned} \langle N'(s), N(s)\rangle &= 0, \\ \langle N'(s), T(s)\rangle = -\langle N(s), T'(s)\rangle &= -k(s), \\ \langle N'(s), B(s)\rangle = -\langle N(s), B'(s)\rangle &= -\tau(s), \end{aligned}$$

so that

$$N'(s) = -k(s)T(s) - \tau(s)B(s)$$

for each $s \in I$. The three equations we have deduced,

$$\begin{aligned} T'(s) &= k(s)N(s), \\ N'(s) &= -k(s)T(s) - \tau(s)B(s), \\ B'(s) &= \tau(s)N(s) \end{aligned}$$

are the *Serret-Frenet equations of the curve* α.

Fundamental Theorem of Local Theory of Space Curves

Let $I \subset \mathbb{R}$ be an open interval and let $k_0, \tau_0 : I \to \mathbb{R}$ be two differentiable functions with $k_0(s) > 0$, for each $s \in I$. Then there exists a curve $\alpha : I \to \mathbb{R}^3$ p.b.a.l. such that $k_\alpha(s) = k_0(s)$ and $\tau_\alpha(s) = \tau_0(s)$ for each $s \in I$, where k_α and τ_α are the curvature and torsion functions of α. Furthermore, α is unique up to a direct rigid motion of Euclidean space $\mathbb{R}^3$.

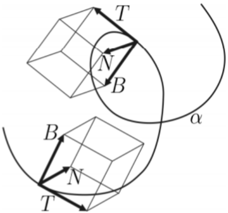

Figure 3: The triplet of vectors associated with each point s of the curve is called the Frenet trihedron of α at s.